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Improved teaching material based on piloting

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The role of the derivative in real life

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Contents

1. Theoretical background ... 3
2. Applications ... 3
   2.1. Applications in physics. Velocity and acceleration ... 3
   2.1. Applications in economics. Marginal revenue. Profit maximization. ... 4
3. Word Problems on the Role of the Derivative in Real Life ... 7
   1. Population Growth ... 7
   2. Speeding Car ... 7
   3. Leaky Tank ... 7
   1. Tank Drainage ... 7
   2. Cooling Coffee ... 7
   3. Rocket Launch ... 7
4. Online Aplication on the Role of the Derivative in Real Life ... 8
5. Reference: ... 9

1. Theoretical background

Let $f: D \to R$ be a real function, where D is an interval or an union of intervals within **R**.
Definition 1 ([Gan97]). We say that function f admits a derivative in $x_0 \in D$ if the limit

1. $$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists in $\overline{R}$. In this case the limit is denoted with $f'(x_0)$ and it is called the derivative of the function f with respect to x_0.

Therefore,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Definition 2 ([Gan97]). Function f is said to be differentiable at $x_0 \in D$ if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists in R (exists and it is finite). In this case the limit is denoted by $f'(x_0)$ meaning

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Observation 1 ([Gan97]) If function f admits a finite derivative in $x_0 \in D$, then it can be interpreted as the slope of the tangent line at $A = (x_0, f(x_0))$ on the graph of the function f. In this case the equation of the tangent is given by:

$$y - f(x_0) = f'(x_0)(x - x_0).$$

Observation 2 ([Gan97]) If the derivative of function f is infinity the equation of the tangent is given by:

$$x = x_0.$$

2. Applications

2.1. Applications in physics. Velocity and acceleration

Exercise 2.1.1. The flight path of a soccer ball is described by the function $x(t)$, where t is in seconds and x is meters above ground:

$$x: R_+ \to R, x(t) = -t^2 + 3t + 4.$$

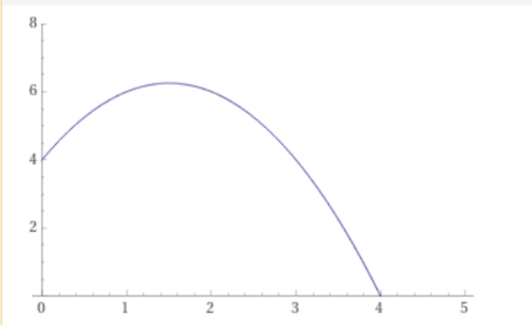
What time will the soccer ball touch the ground?
From what distance above the ground was the soccer ball launched?
What is the velocity of the soccer ball at the end of 2 seconds?
What is the maximum height the soccer ball will reach and at what time will this occur?
What is the acceleration (with direction) of the soccer ball at t=3 s?

Solution.
The soccer ball will touch the ground when $x(t) = 0$, because x stands for the distance above ground. Then, we have:

$-t^2 + 3t + 4 = 0,$

with the roots $t = -1$ and $t = 4$. The time cannot be negative and t=4 seconds and the soccer ball hits the ground, after 4 seconds after his launch.

In the following figure we can visualize the flight path of the soccer ball (www.wolframalpha.com):



Substituting $t = 0$ in the function x we can obtain the meters above the ground, when the soccer ball was launched:
$x(0) = 4\,m.$
The instantaneous velocity at a given time is the derivative of a function that represents the position at time t. Then, the velocity of the soccer ball at the end of t seconds is:

$$v(t) = x'(t) = -2t + 3,$$

And at the end of 2 seconds is v(2)=-1

In order to find the maximum height, the soccer ball will reach and at what time will this occur, we have to find the time when the instantaneous velocity at that time is zero: $v(t) = 0$.

where $a = -1, b = 3, c = 4$. Then, the time when the soccer ball reaches the maximum height is 3 s and the maximum height is 6.25 m.

2.1. Applications in economics. Marginal revenue. Profit maximization.

Exercise 2.2.1. The price of a good, sold by a firm, is given by the function:

$$p: R \to R, p(x) = -\frac{x}{2} + 20,$$

where x is the amount produced by firm. What is the total revenue of the firm? What is the marginal revenue function of the firm? Calculate the price, the total revenue and marginal revenue for the quantities x=18, x=20, x=22, respectively. What do you notice?
Solution. The total revenue of the firm, denoted by TR, is obtained as the price times the sold quantity and

$$TR: R \to R, TR(x) = -\frac{x^2}{2} + 20x.$$

The marginal output, denoted by MR, is the change in revenue resulting from a change in quantity, that means:

$$MR(x) = TR'(x) = -x + 20.$$

For $x = 18$, we obtain: $p(18) = 11, TR(18) = 198, MR(18) = 2.$
For $x = 20$, we obtain: $p(20) = 10, TR(20) = 200, MR(20) = 0.$
For $x = 24$, we obtain: $p(24) = 8, TR(22) = 192, MR(22) = -4.$

We can notice that the maximum value of the profit is 200, for the marginal revenue 0. A manager who wants to maximize profits must analyze how the increasing of the output would affect the price, the marginal revenue and the total revenue.

Exercise 2.2.2. A manager of a firm, based on a set of real-world data, estimates the profit of the firm as the function:

$$P: R \to R, P(x) = -\frac{1}{3}x^3 - 3x^2 + 27x - 10,$$

where x is the quantity of the good sold. Find the output that should be produced in order to maximize the profit.

Solution. Let x^* be the amount of good that should be produced. If more or less than this

If more or less than this quantity was produced, the profit would not be maximized. From the geometric point of view, the derivative of the profit function is the slope of the curve. If we want that x^* to maximize the profit, then the slope is positive to the left of $x^*$and negative to the right of x^*:

$$\begin{cases} P'(x) > 0 & x < x^* \\ P'(x) < 0 & x > x^* \end{cases}$$

At the point x^*, the slope of is 0. Hence, the output x^* should be chosen such that the following condition:

$$\left.\frac{dP(x)}{dx}\right|_{x=x^*} = P'(x)|_{x=x^*} = 0$$

(2)

holds.

The condition (1) is a necessary condition, but not a sufficient one. If a bit more or a bit less quantity than x^* is produced, the available profit must be smaller than that corresponding to x^* . Otherwise, another output is better than x^*. As (1) is satisfied, at x^*, $P'(x)$ must be decreasing, that means the derivative of $P'(x)$ must be negative at x^*.

Therefore, the sufficient condition for x^* to maximize the profit function is:

$$P''(x)|_{x=x^*} < 0.$$

In our particular case, we have:

$P'(x) = 0,$

or equivalent to the second-degree equation:

$$-x^2 - 6x + 27 = 0,$$

with the solutions -9 and 3. From economic point of view, the quantity cannot be negative, then we consider only $x = 3$.

In order to verify if this value leads to the maximum profit, we have to compute the second derivative of P:

$$P"(x) = -2x - 6$$

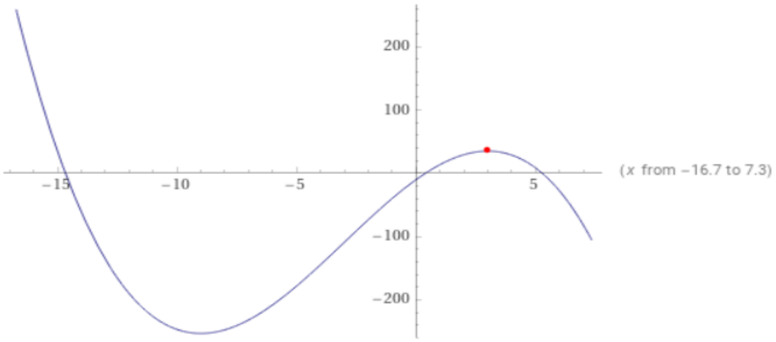
and $P"(3) = -12 < 0$.

The manager should produce 3 units of product in order to obtain a maximum profit given by $P(3) = 35$ monetary units.

Using Wolframalpha we can visualize the graph of the profit and the maximum point and the corresponding profit.

$$\max\left\{-\frac{x^3}{3} - 3x^2 + 27x - 10 \,\middle|\, x > 0\right\} = 35 \text{ at } x = 3$$

Plot



3. Word Problems on the Role of the Derivative in Real Life

A.

1. Population Growth

The population of a small town is modeled by the function P(t) = 1000e^0.05t, where t represents the number of years since the population was first recorded. Find the rate at which the population is growing after 10 years, and interpret the meaning of this rate in the context of the problem.

2. Speeding Car

A car is traveling along a straight road. The position of the car at time t is given by the equation $s(t) = 4t^2 - 16t + 20$, where s(t) represents the distance from the starting point in meters. Find the velocity of the car when t = 3, and interpret the meaning of this velocity in the context of the problem.

3. Leaky Tank

A water tank in the shape of an inverted cone has a radius of 5 meters and a height of 10 meters. The water level in the tank is decreasing at a constant rate of 2 meters per hour. Find the rate at which the volume of water in the tank is decreasing when the water level is 6 meters, and interpret the meaning of this rate in the context of the problem.

B.

1. Tank Drainage

A cylindrical tank with a radius of 5 meters is being drained at a constant rate of 2 cubic meters per minute. The height of the water in the tank is initially 10 meters. Find the rate at which the water level is decreasing when the height of the water is 6 meters, and interpret the meaning of this rate in the context of the problem.

2. Cooling Coffee

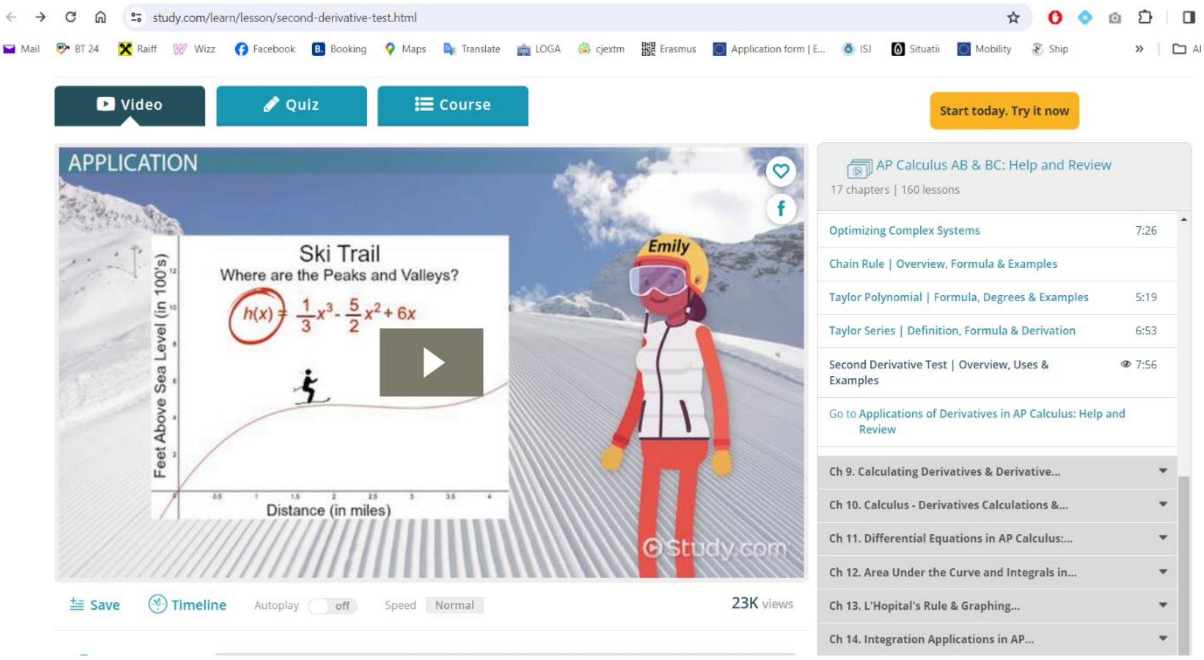
A cup of coffee is left at room temperature (20 degrees Celsius). The temperature of the coffee is modeled by the function $T(t) = 80e^{-0.1t} + 20$, where T(t) represents the temperature of the coffee in degrees Celsius after t minutes. Find the rate at which the temperature of the coffee is decreasing after 5 minutes, and interpret the meaning of this rate in the context of the problem.

3. Rocket Launch

A rocket is launched vertically into the air. The height of the rocket above the ground at time t is given by the equation $h(t) = 100t - 5t^2$, where h(t) represents the height in meters. Find the velocity of the rocket when t = 4 seconds, and interpret the meaning of this velocity in the context of the problem.

4. Online Aplication on the Role of the Derivative in Real Life

1.



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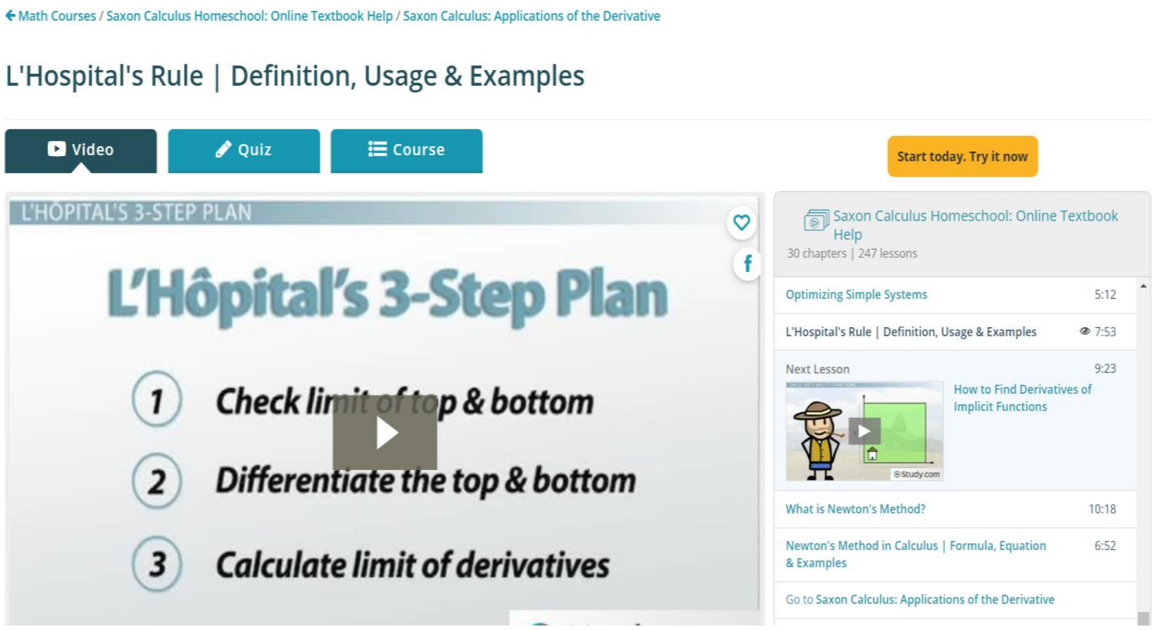

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