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Colegiul Național „Constantin Diaconovici Loga" din Timișoara

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Financial Mathematics

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1. Theoretical background

1.1. Percentage

DEFINITION

Percentage is defined as a given part or amount in every hundred. It is a fraction with 100 as the denominator and is represented by the symbol "%".

A percentage is a dimensionless number (pure number); it has no unit of measurement.

FORMUA

$$\frac{part}{whole} = \frac{\%}{100}$$

- Find: $\text{p \% of A} = \frac{\text{p}}{100} \cdot \text{A} = \text{B}$
- Find what percent of A is B: $\text{p} = \frac{\text{B}}{\text{A}} \cdot 100$
- How to find A if P percent of it is B: $\text{A} = \frac{\text{B}}{\text{p\%}} = \frac{\text{B} \cdot 100}{\text{p}}$

1.2. Simple interest

DEFINITION

Simple interest is the cost of borrowing money without accounting for the effects of compounding. In other words, simple interest only applies to the principal amount.

Simple interest is a fixed percentage.

FORMULA

$$I = P \times r \times t$$
$$A = P + I = P + P \cdot r \cdot t = P \cdot (1 + r \cdot t)$$

I = Interest,
P = principal, starting amount,
r = interest rate in decimal form,
t = time,
A = end amount: principal plus interest.

1.3. Compound interest

DEFINITION

Compound interest is the addition of interest to the principal sum of a loan or deposit, or in other words, interest on principal plus interest.

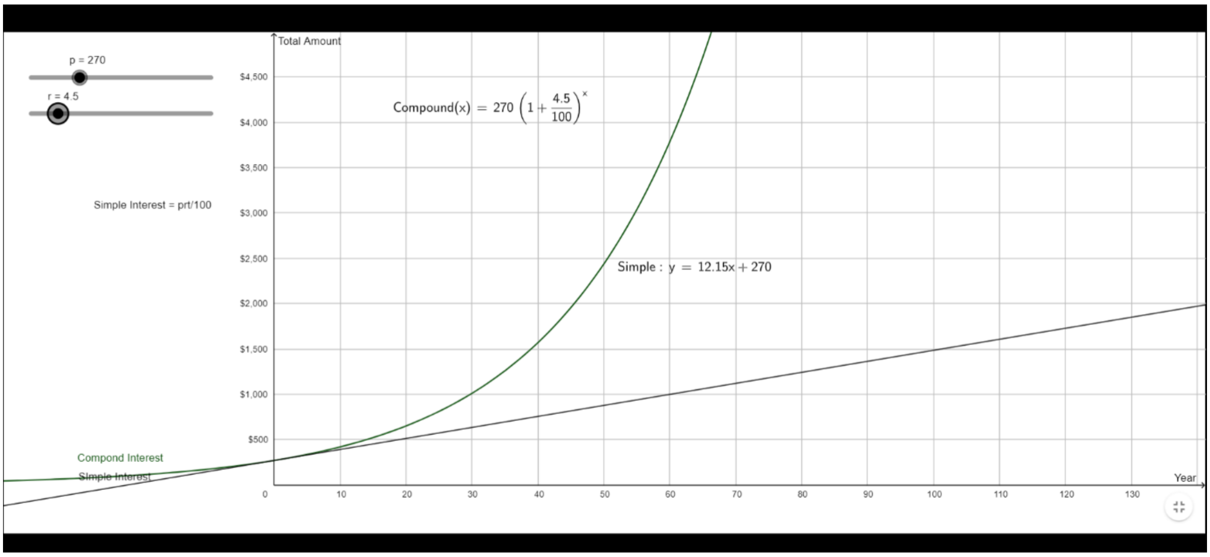
Compound Interest = Future Value of P

FORMULA

$$Future\ Value = P \times \left(1 + \frac{r}{n}\right)^{nt}$$

P = principal,
r = interest rate,
t = time in years,
n = number of times per year interest is compounded

Simple vs Compound Interest
https://www.geogebra.org/classic/c37PDtmt



1.4. Annual Percentage Rate (APR)

Annual Percentage Rate (APR): Rates of interest only have meaning when they are related to a time interval. Rates of interest, expressed above, giving an actual rate of interest over a stated interval of time, are effective rates of interest. Where the effective rate of interest is expressed as a fraction of a year ($1/p$) it may be converted to an annual rate by multiplying by p. thus, 3% per quarter would be quoted as '12% per annum, converted quarterly'. Interest rates quoted in this way are known as nominal rates of interest. Quoted interest rates on savings products offered by financial institutions are often nominal rates, e.g. converted half yearly. Corresponding to a nominal rate of interest, there exists an effective annual rate of interest.

If a person invests €100 for one year at 10% per annum, convertible half-yearly, (effective rate of interest of 5% per half year) – not the same as an effective rate of interest of 10% pa - the amount at the end of the year is

$$€100\,(I + \frac{0.10}{2}) \cdot 2 = 1.1025.$$

If the annual rate of interest is i% pa the amount at the end of the year is
€100 $(I + r)$.
Therefore, $1 + r = 1.1025$, giving $r = 10.25\%$.
A rate of interest expressed as 10% pa, convertible half yearly, is the same as an effective rate of interest of 10.25% or quoted as the APR (annual percentage rate).

This process may be generalized as follows:

Nominal rate compounded n times per year: $S = Po(1 + \frac{r}{n})^{nt}$

APR rate compounded annually: $S = Po(1 + APR)^t$

Since the yield is the same, $Po(1 + \frac{r}{n})^{nt} = Po(1 + APR)^t$ giving an APR of $(1 + \frac{r}{n})^n - 1$.

1.5. The exponential function

Definiție. The function $f:\boldsymbol{R}\to(0,+\infty)$, $f(x) = a^x$, where $a > 0$, $a \neq 1$ is called ***exponential function whit baze a***.

Proprieties:

1). a). If $a > 1$, then for $x > 0$ have $a^x > 1$ have $a^x > 1$, and for $x < 0$ we have $a^x < 1$.

b). If $0 < a < 1$, then for $x > 0$ have $a^x < 1$, and for $x < 0$ we have $a^x > 1$.

2). If $x = 0$. atunci oricare ar fi $a > 0$ are loc $a^0 = 1$

3). For $a > 1$, the exponential function $f:\boldsymbol{R}\to(0,+\infty), f(x) = a^x$ is strictly increasing, and for $0 < a < 1$, the function is strictly decreasing.

4). The exponential function $f:\boldsymbol{R}\to(0,+\infty), f(x) = a^x, a > 0, a \neq 1$ is bijective.

The graph of the exponential function

The graph of the exponential function is built by points.

Example.

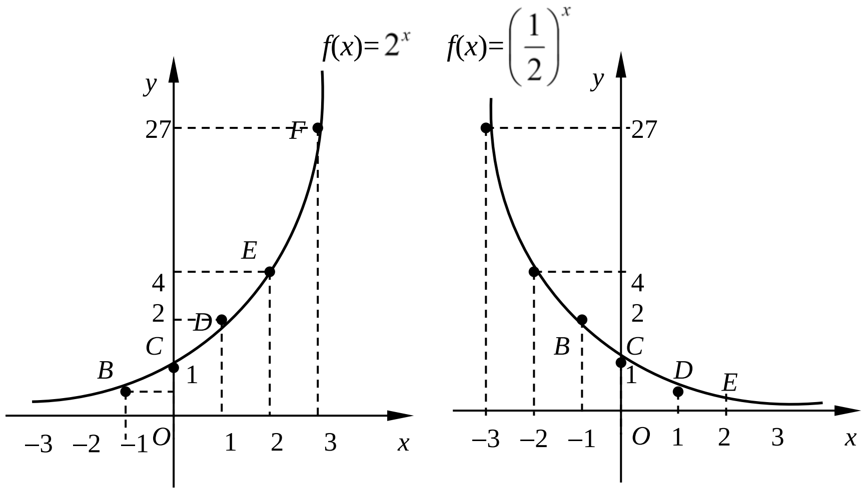
Construct the graph of the function f:**R**→(0,+∞), $f(x) = a^x$, for $a \in \left\{2, \frac{1}{2}\right\}$.

A table of values is drawn up for the two cases:

x	$-\infty$	-3	-2	-1	0	1	2	3
$f(x)$		$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

x	$-\infty$	-3	-2	-1	0	1	2	3
$f(x)$		-27	-4	-2	1	$\frac{1}{2}$	4	27

The graphs of the two functions are represented below:



2. Application of Financial Math

2.1. Investments with simple interest versus compound interest

1. Invest €500 that earns 10% interest each year for 3 years, where each interest payment is reinvested at the same rate:

Table 1 Development of Nominal Payments and the Terminal Value, *S*.

	Nominal Interest	*S*
Year 1	50	550 = (1.1)
Year 2	55	605 = 500(1.1)(1.1)
Year 3	60.5	665.5 = $500(1.1)^3$

2. A principal of €25000 is invested at 12% interest compounded annually. After how many years will it have exceeded €250000?

$10P = P \cdot (1+r)^n$
$250{,}000 = 25{,}000 \cdot (1.12)^n$
$10 = 1.12^n$
$\ln 10 = n \cdot ln 1.12$
$\frac{ln 10}{ln 1.12} = n \approx 20.3177$

Compounding can take place several times in a year, e.g. quarterly, monthly, weekly, continuously. This does not mean that the quoted interest rate is paid out that number of times a year!

3. Assume the €500 is invested for 3 years, at 10%, but now we compound quarterly:

Table 2 Quarterly Progression of Interest Earned and End-of-Quarter Value, *S*.

Quarter	Interest Earned	*S*
1	12.5	512.5
2	12.8125	525.3125
3	13.1328	538.445
4	13.4611	551.91

Generally: $S = P \cdot \left(1 + \frac{r}{m}\right)^{n\,m}$ where *m* is the amount of compounding per period *n*.

4. €10 invested at 12% interest for two years. What is the future value if compounded
a) annually ? b) semi-annually ? c) quarterly?
d) monthly ? e) weekly?

As the interval of compounding shrinks, i.e. it becomes more frequent, the interest earned grows. However, the increases become smaller as we increase the frequency. As compounding increases to continuous compounding our formula converges to:

$$S = P \cdot e^{rt}$$

5. A principal of €10000 is invested at one of the following banks:
a) at 4.75% interest, compounded annually
b) at 4.7% interest, compounded semi-annually
c) at 4.65% interest, compounded quarterly

d) at 4.6% interest, compounded continuously

Which is the best bank to lodge the money?
=>
a) $10{,}000(1.0475) = 10{,}475$
b) $10{,}000(1+0.047/2)^2 \approx 10{,}475.52$
c) $10{,}000(1+0.0465/4)^4 \approx 10{,}473.17$
d) $10{,}000e^{0.046t} \approx 10{,}470.74$

6. Determine the annual percentage rate, APR, of interest of a deposit account which has a (simple) nominal rate of 8% compounded monthly.

$$\left(1+\frac{0.08}{12}\right)^{1\cdot 12} \approx 1.083$$

7. A firm decides to increase output at a constant rate from its current level of €50000 to €60000 over the next 5 years. Calculate the annual rate of growth required to achieve this growth.

$50000(1+r)^5 = 60000$
$(1+r)^5 = 1.2$
$1+r = \sqrt[5]{1.2}$
$r \approx 3.7\%$

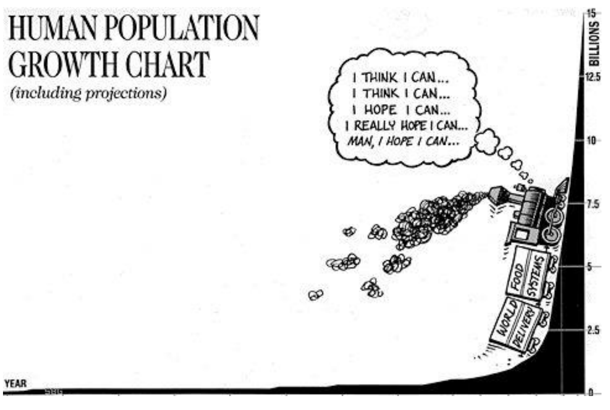
2.2. Exponential growth

10 Real Life Examples Of Exponential Growth

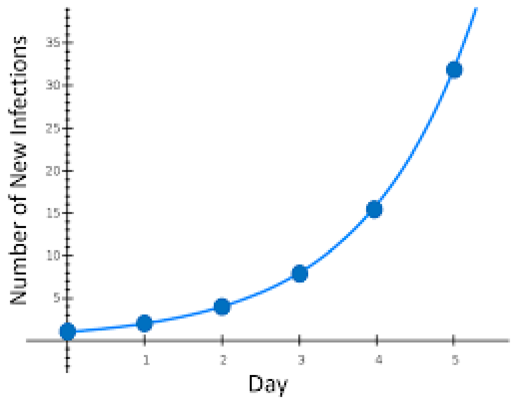
- Microorganisms in Culture.
- Spoilage of Food

When we keep cooked or uncooked food at room or warm temperature, it begins to get spoiled after some time. Almost everyone has come across the green discolouration which ruins the food and spreads quite fast.

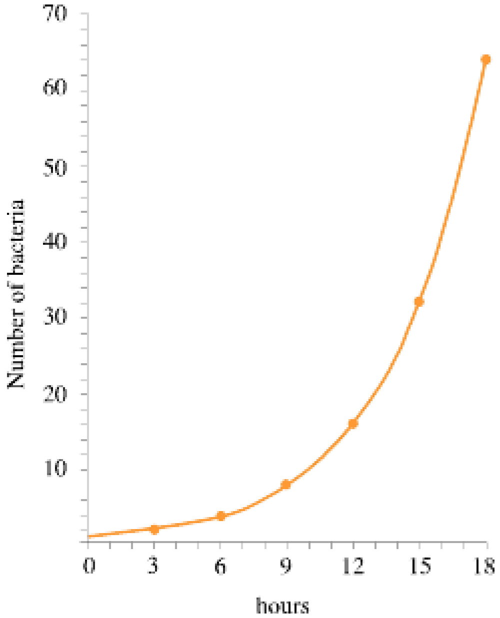
- Human Population. .



- Pandemics.



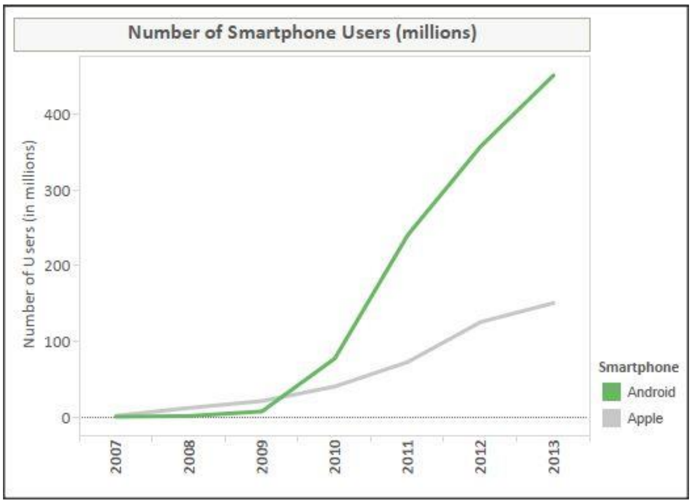
- Invasive Species.

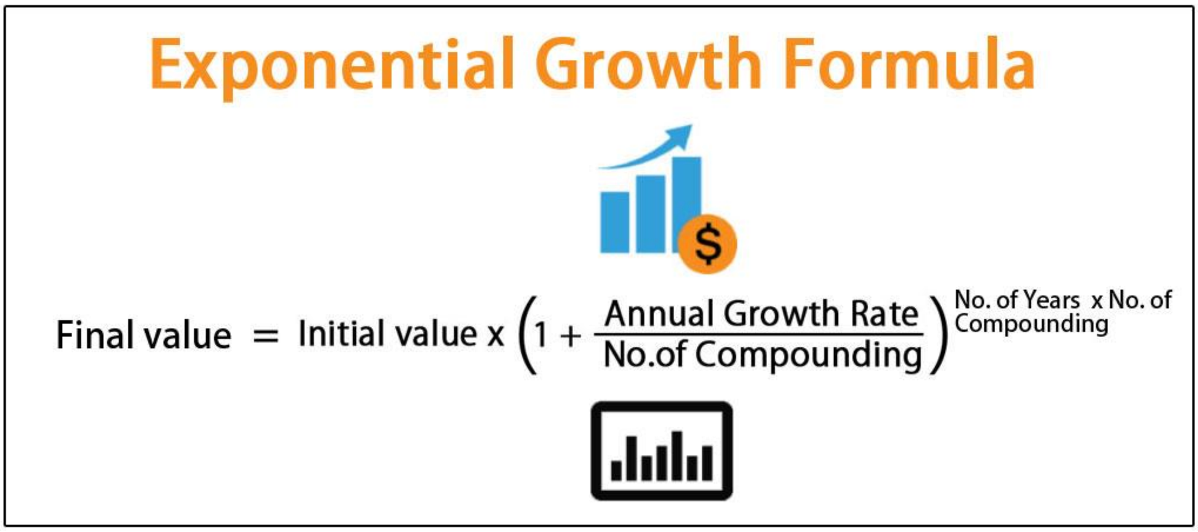


Exponential Growth

- Ebola Epidemic.
- Fire.
- Cancer cells
    - Compound Interest

Smartphones Uptake and Sale

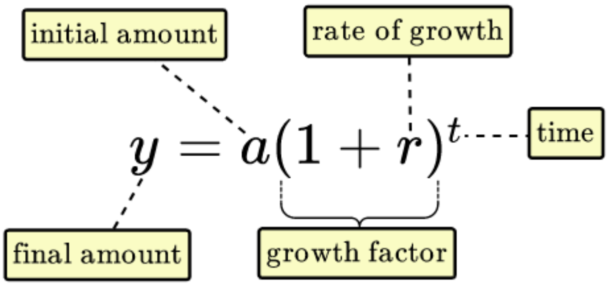




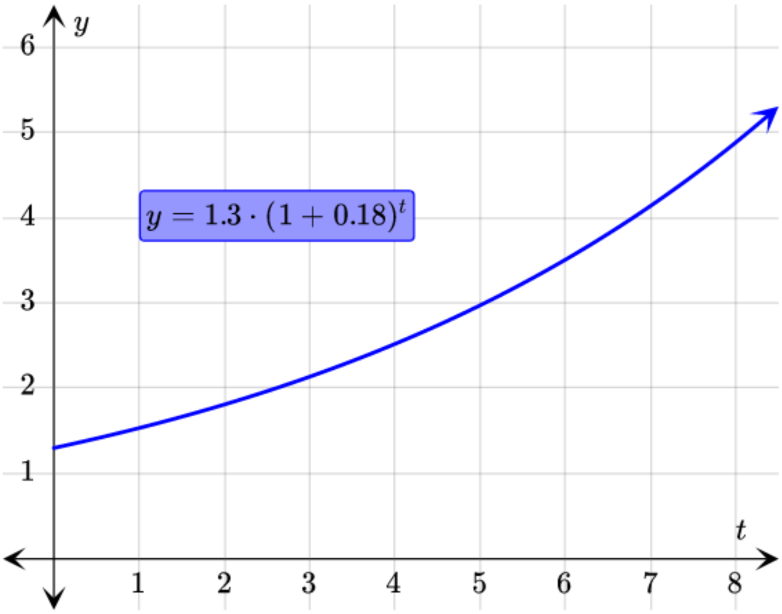
Exponential growth occurs when a quantity **increases** by the same factor over equal intervals of time. This leads to an exponential function, where the independent variable in the exponent, t, is time.

$$y = a \cdot b^t$$

Since the quantity **increases** over time, the constant multiplier b has to be **greater** than 1. Thus, the *growth factor* b can be expressed as $b = 1 + r$, where r is some positive number. The resulting function is called an *exponential growth function*.



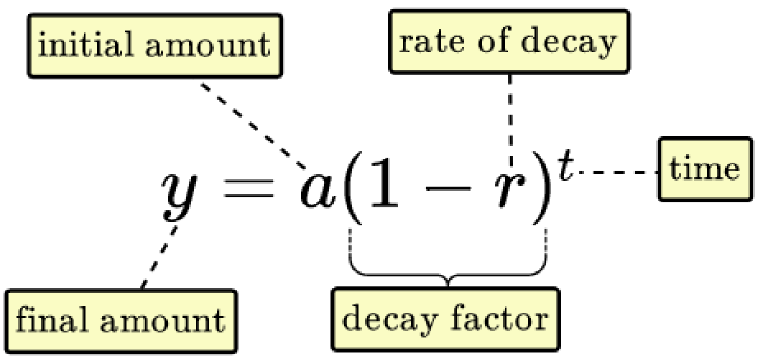
The constant r can then be interpreted as the *rate of growth*, in decimal form. A value of 0.06, for instance, means that the quantity increases by 6% over every unit of time. As is the case with all exponential functions, a is the y –coordinate of the $y - intercept$.



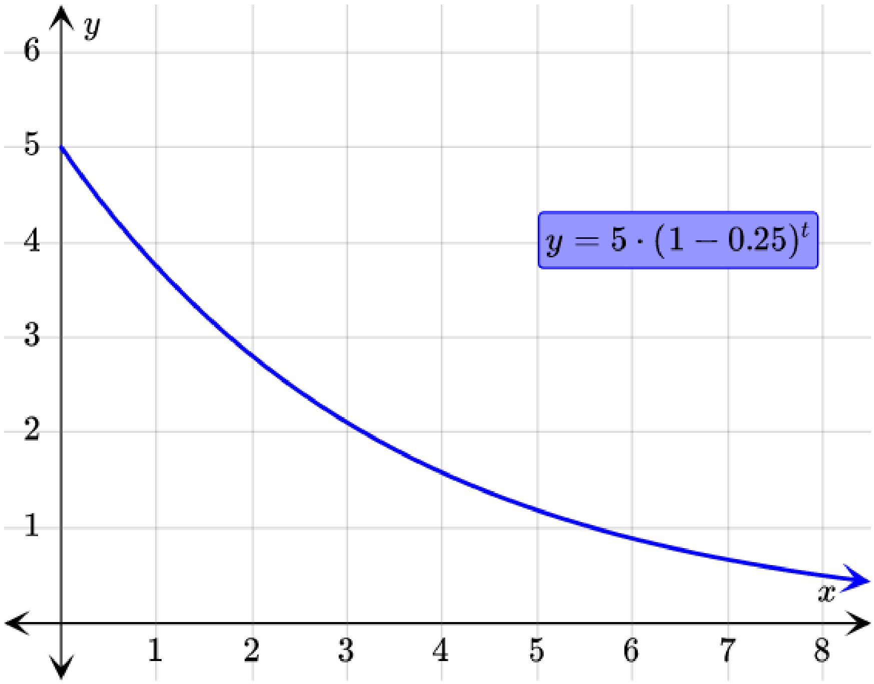
Since the growth factor is greater than 1, the quantity grows faster and faster, without bound.

2.3. Exponential Decay

The counterpart of exponential growth is exponential decay; when a quantity **decreases** by the same factor over equal intervals of time, the constant multiplier of the *exponential decay function* is less than 1. This factor can be expressed as $(1 - r)$ and is known as the *decay factor*.



The constant r can then be interpreted as the *rate of decay*, in decimal form. A value of 0.12, for instance, would mean that the quantity decreases by 12% over every unit of time.



Since the decay factor is smaller than 1, the quantity decays toward 0 over time.

Since the decay factor is smaller than 1, the quantity decays toward 0 over time.

As an example, a growth of 10% every unit of time gives the growth factor

$$(1 + r) = (1 + 0.1) = 1.1.$$

Similarly, it can be necessary to use the growth or decay factor to find the rate of growth or decay. For instance, a factor of 0.85 indicates a decay. Thus, r can be found by equating 0.85 with the decay factor $(1 - \mathrm{r})$, and solving the equation.

$$(1 - r) = 0.85 \Leftrightarrow r = 0.15$$

EXAMPLE

Graph the exponential growth or decay function

During a time period, the number of carps in a small lake can be modeled by the function

$$H(t) = 800 \cdot 0{,}88^t,$$

where t is the time in years. State whether the function shows growth or a decay, and then find the rate of growth or decay, r. Finally, graph the function.

Solution

To begin, let's analyze the given function rule. It's written in the form $y = a \cdot b^x$, where a is the initial value and b is constant multiplier/growth factor.

The constant multiplier, 0.88, is less than 1, so it is a decay factor. Therefore, the function shows decay. Since the decay factor is always equal to $1 - r$, we can write the equation.

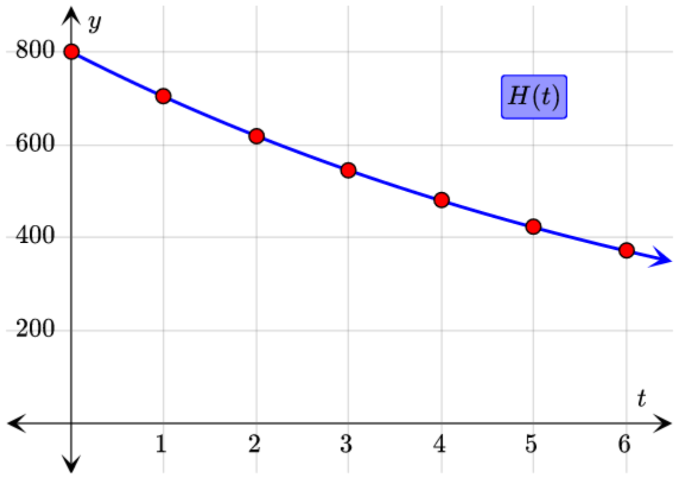
$$0.88 = 1 - r,$$

which can be solved for r.

$$0.88 = 1 - r$$
$$LHS + r = RHS + r$$

$$0.88 + r = 1$$
$$LHS - 0.88 = RHS - 0.88$$
$$r = 0.12$$

Thus, the rate of decay is $0.12,\ or\ 12\%$, per year. The initial value is 800, and the constant multiplier is 0.88. Using this information, we can graph the exponential decay function by plotting some points that lie on H and connecting them with a smooth curve.



Examples of Exponential Decay

1. Radioactive Decay
2. Reselling Cost of a Car
3. Population Decline
4. Treatment of Diseases
5. Consuming a Bag of Candy
6. Radiocarbon Dating
7. Calculating the amount of drug in a person’s body
8. Healing of Wounds

2.4. The monthly interest rate

One savings account, with a principal of $100, offers an annual interest rate of 15% compounded twice a year. Find the balance in the account after 5 years.
Another savings account with the same principal will have the same balance after 5 years. However, the interest is compounded monthly. Find the interest rate of the second account.

Solution

First, we'll find the function rule describing the growth of the first account. It is given that $P = 100, r = 0.15,$ and $n = 2$. Substituting these values in the compound interest formula gives

$B(t) = 100\left(1+\frac{0{,}15}{2}\right)^{2t}$. Let's simplify this function before continuing.

$B(t) = 100(1+0{,}075)^{2t} = 100 \cdot 1{,}075^{2t}$

Since the interest will accrute for 5 years, $t = 5$. Therefore, we can find the account balance by evaluating $B(5)$.
$B(5) = 100 \cdot 1{,}075^{10} = 206{,}10315 \cong 206{,}10$

The account balance is $206.10 after 5 years. Now we can consider the second account. We know the balance of both accounts is equal, at least when both just had their interest compounded. This means that B(t) also describes the growth in the second account. However, since the interest in the second account accrues monthly, or 12 times a year, n=12. Thus, the exponent in the rule should be 12t. By using the equality

$$2t = \frac{1}{6} \cdot 12\,t$$

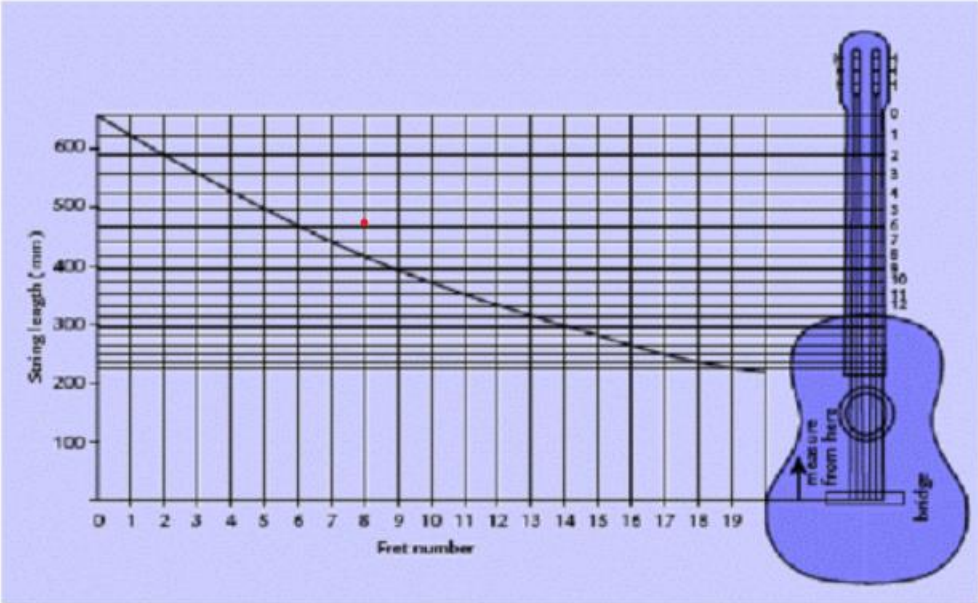
and the power of a power property, we can rewrite B(t) so that it's possible to find the monthly interest rate. Having the exponent 12t means that the base of the power is equal to the monthly growth factor, which we can then use to find the monthly interest rate.

$$B(t) = 100 \cdot 1{,}075^{2t} = 100 \cdot 1{,}075^{\frac{1}{6}\cdot 12t} = 100 \cdot (1{,}01212\ldots)^{12t}$$
$$\cong 100 \cdot 1{,}012^{12t}$$

We find an approximate monthly growth factor 1.012, which corresponds to a rate of growth that is 0.012. Thus, the monthly interest rate is roughly 1.2%.

2.5. Exponential decay on a guitar

Practical Activity
This activity uses real data of the lengths of string on a guitar for different notes, to show exponential decay in a real world situation. The graphs may be explored using a spreadsheet and/or graphics calculator. The data might be usefully compared to the same distances on a guitar in your classroom.
The frets on a guitar are placed so that they make the correct vibrating string length for the note of music. We are interested in how the vibrating string length changes for each fret position. The function is such at after 12 exponential reductions (one per fret) the string length reduces to half its open length.
$b^{12} = 0{,}5 \Rightarrow b = \sqrt[12]{0{,}5} \cong 0{,}914$



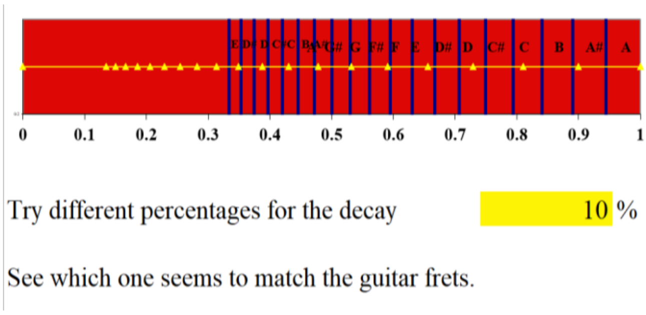
Spreadsheet: Guitar frets

The [spreadsheet (Excel - 114Kb)]() offers three versions of the function:

as a percentage decay (for example, 10% means the base number is 0.9) as a function, where students try different base numbers, e.g. 0.9. as a fraction, where students enter different common fractions less than 1 and see whether or not they come close to a fret position. Some of them do, and the discoverer of this remarkable fact was Pythagoras.

Instructions

Open the spreadsheet. Choose one of the three options using the tabs at the base. Try different numbers in the yellow box. The graph will adjust appropriately.



3. Reference

[1] https://studiousguy.com/

[2] *Math in Society* (on OpenTextBookStore) by David Lippman, and is used under a CC Attribution-Share Alike 3.0 United States (CC BY-SA 3.0 US) license.

[3] *Math for the Liberal Arts* (on Lumen Learning) by Lumen Learning, and is used under a *CC BY: Attribution* license.

[4] http://smartvic.com/teacher/mdc/structure/St55002P.html