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Co-funded by the Erasmus+ Programme of the European Union

Project title: **Computer supported Math teaching,** COMPMATH
Ref. No.: KA210-SCH-ABDA6A90
Action Type: KA210-SCH - Small-scale partnerships in school education

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"Origin of the notion of derivate" the mathematical modeling of a physical problem

2022/2023

Contents

1. Theoretical background

1.1. Origin of the notion of derivative

In mathematics, the derivative of a function is one of the fundamental concepts of mathematical analysis, along with the primitive (inverse derivative or anti-derivative).

The derivative of a function at a point signifies the rate at which the value of the function is changed when the argument is changed. In other words, the derivative is a mathematical formulation of the notion of rate of variation. The derivative is a very versatile concept that can be viewed in many ways. For example, referring to the two-dimensional graph of the function f, the derivative at a point x represents the slope of the tangent to the graph at the point x. The slope of the tangent can be approximated by a secant. With this geometric interpretation, it is not surprising that derivatives can be used to describe many geometric properties of function graphs, such as concavity and convexity.

It should be noted that not all functions admit derivatives. For example, functions have no derivatives at points where they have a vertical tangent, at points of discontinuity and at points of return.

1.2. The Leibnitz–Newton dispute

Differential and integral calculus were invented virtually simultaneously, but independently of each other, by the Englishman Isaac Newton (1643–1727) and the German mathematician Gottfried Wilhelm von Leibniz (1646–1716).

It can be mentioned, with the almost anecdotal title, but absolutely real, that the scientific world of that moment (1685-1690) witnessed, almost "heartily", for several years, an open and permanent dialog between the two titans, Leibnitz and Newton. Only after the two scientists have come to an understanding of the concepts and concepts from both points of view (the physicist and the mathematician), after agreeing with the preliminary notions, the limits and the methodology of approaching concepts, etc., the two of them were able to explain to the rest of the scientific world what it was all about.

1.3. Derivative and derivability

The derivative arose from the need to express the rate at which a quantity y changes (varies) as a result of the change (variation) of another quantity x to which it is bound by a function. Using the symbol Δ to note the change (variation) of a quantity, this rate is defined as the limit of the ratio of variations (differences):

$$\frac{\Delta y}{\Delta x}$$

As Δx tends toward 0 or otherwise expressed Δx was in the vicinity of 0. In Leibniz's notation, the derivative of y in relation to x is written

$$\frac{dy}{dx}$$

suggesting the ratio of two infinitesimal numerical differences (quantities) (in the vicinity of 0). The expression above can be pronounced either "dy supra dx" or "dy la dx".

In contemporary mathematical language, no reference is made to quantities that vary; the derivative is considered a mathematical operation on functions. The formal definition of this operation (which no longer makes use of the notion of infinitesimal quantities) is given by the limit when h tends to 0 (e in the vicinity of 0) of the following expression:

$$\frac{f(x+h)-f(x)}{h}$$

1.4 The role of the derivative in physics

There were two problems, one physical - the mathematical modeling of the intuitive notion of the speed of a mobile - and the other geometric - the tangent to a flat curve -, which led to the discovery of the notion of derivative. We have used several times references to the speed of a mobile, but only now will we be able to give the mathematical definition of this concept.

a) *Instantaneous speed of a mobile*. We assume that on an axis Δ that a mobile moves in the positive direction of the axis and at the time t the mobile is at the abscissa point s(t). If the motion is uniform, then for any two moments t_1, t_2 ($t_1 \neq t_2$) the ratio $\frac{s(t2)-s(t1)}{t2-t1}$ is constant, equal to the speed *v* of the mobile; in this case, it is known that $s(t) = v \cdot t$. But what happens if the mobile no longer has a uniform movement, although it moves along the same axis Δ?

The previous ratio will no longer be constant and for any moment t_1, t_2 $(t_1 \neq t_2)$ the ratio $\frac{s(t2)-s(t1)}{t2-t1}$ between the distance traveled and the elapsed time is called the *average speed* of the mobile between the respective moments (it should be noted that we did not fix an ordering of moments t_1, t_2). Let's now consider a reference moment t_0, practically there are no uniform movements, but on smaller and smaller intervals the movement tends to become uniform. For $t \to t_0$, $t \neq t_0$ it can be considered that the movement of the mobile during the time interval between t_0 and t tends to become uniform, and the respective average speed tends to a characteristic of the movement exactly at the moment t_0. This suggests the definition of the instantaneous speed of the mobile at time t_0 as the limit:

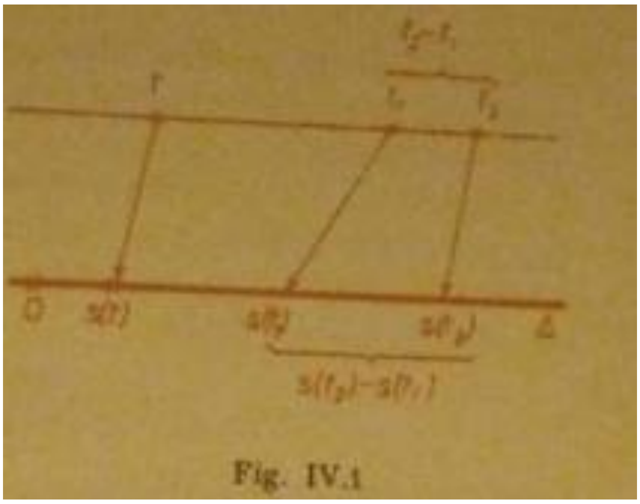


Fig. IV.1

$$v(t_0)=\lim_{t\to t0}\frac{s(t)-s(t0)}{t-t0}$$

assuming that this limit exists.

So, $v(t_0)$ is the limit for $t \to t_0$, of the average speed of the mobile between the moment t_0 and $t \neq t_0$

For example, in the study of free fall, it was proven that the space traveled in meters after t seconds is $s(t) = \frac{1}{2}gt^2$. Fixing a certain moment t_0, the speed at time t_0 is

$$v(t_0)=\lim_{t\to t0}\frac{\frac{1}{2}gt^2-\frac{1}{2}gt_0^2}{t-t0} = \frac{1}{2}\lim_{t\to t0} g(t+t0) = \frac{1}{2}g\cdot 2t_0 = g\cdot t_0$$

and the speed at any moment t will be

$$v(t) = g \bullet t$$

(g is the gravitational acceleration, $g = 9.81\ m/s^2$).

Similarly, if v(t) is the speed of the mobile at any time t, then the acceleration of the mobile at time t_0 is defined as

$$a(t)=\lim_{t\to t0}\frac{v(t)-v(t0)}{t-t0}$$

Assuming that this limit exists. Newton's fundamental law of mechanics shows that at each moment t the force F(t) acting on the mobile, the mass m of the mobile and the acceleration a(t) are given by the relation $F(t) = m \cdot a(t)$

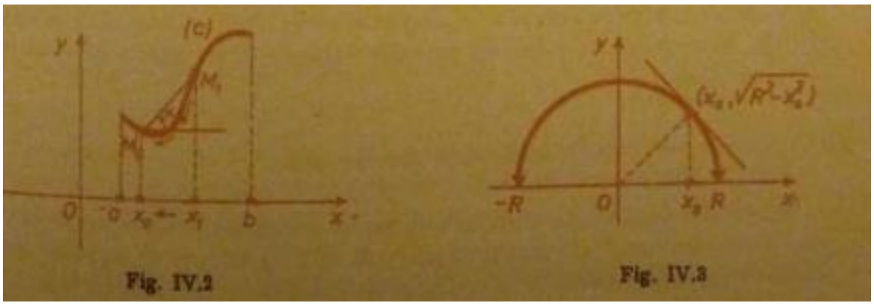


Fig. IV.2 Fig. IV.3

2. Direct application of the derivative in physics

2.1 Velocity and acceleration of a mobile

Consider an axis Δ on which an origin, a direction and a unit of measurement have been fixed. Let it be a mobile (assimilable to a point on Δ); denoting with *s(t)* the abscissa of the point where the mobile is at time t (also called the space traveled by the mobile) and assuming that *s* is a function derivable at a point t_0, then the *instantaneous speed* $v(t_0)$ of the mobile at the time is defined t_0 , as being the derivative of the space in t_0, that is

$$v(t_0)=g'(t_0)$$

If the function g is twice differentiable in t_0, then $g'(t_0) = v'(t_0)$ is called the acceleration of the mobile at time t_0.

We note that in a rectilinear movement the speed is the derivative of the first order, the acceleration the derivative of the second order, of space in relation to time.

These definitions constitute a natural application of definition I.1 to the considered physical model.

Example

We assume that the law of motion of a mobile about one axis is expressed by the relation

$$g(t)=e^{-1}\cos t$$

We want to determine the acceleration of the mobile after 2 seconds and calculate the speed limit as t tends to ∞ .

We have v(t)=g'(t)= -e^{-1}cos t -e^{-1}sin t= -e^{-1}(cos t +sin t) and g(t)=v'(t)= -2e^{-1}sin t_0 so g(2)=$\frac{2sin2}{e^2}$ and

$\lim_{t\to\infty} v(t) = -\lim_{t\to\infty} \frac{cos\, t+sin\, t}{e^t}$ and this limit is zero because $|\frac{cos\, t+sint}{e^t} < \frac{2}{e^t}|$, any t (it is also said that the motion is damped for t -> ∞.)

2.2. The intensity of the electric current

Denoting as Q(t) the charge passing through a section of a conductor in the time interval [0, t], then for any distinct instants t1, t2, the difference Q(t2) - Q(t1) is the charge passed between moments t1, t2, and the quantity $\frac{\mathrm{Q(t2)} - \mathrm{Q(t1)}}{t2-t1}$ is the average load related to the time interval between the two moments. In analogy with the previous mechanical model, if we fix a time t0 and assume that the function Q is differentiable in t_0, then the derivative Q'(t_0) is the rate of variation of the electric charge at time t_0, Q(t_0) = $\lim_{\mathrm{t}\to\mathrm{t0}} \frac{Q(t) - Q(t0)}{t-t0}$. It receives the specific denomination of the intensity of the electric current at time t_0.

If we consider an electric circuit consisting of an inductance L, a capacitance C and a resistance R, the voltage being constant, then from Kirchhoff second law it follows that the intensity of the current through that circuit will verify the following relationship (called differential equation)

$$L_{(t)} + R_{(t)} + C_{(t)} = 0, \quad \forall t.$$

Then there is the problem of determining the function, knowing L, R, C, which requires developments of the mathematical analysis apparatus

2.3. Linear mass density

We consider a supposed material mass distributed on a bar, assimilated as an interval [a,b]. For any point x ∈ [a, b] we denote by m(x) the mass of the portion included in the interval [a, x]. Fixing a point x_0 € (a, b), the ratio $\frac{m(x)-m(x_0)}{x-x_0}$ represents the average density between the points x, x_0.

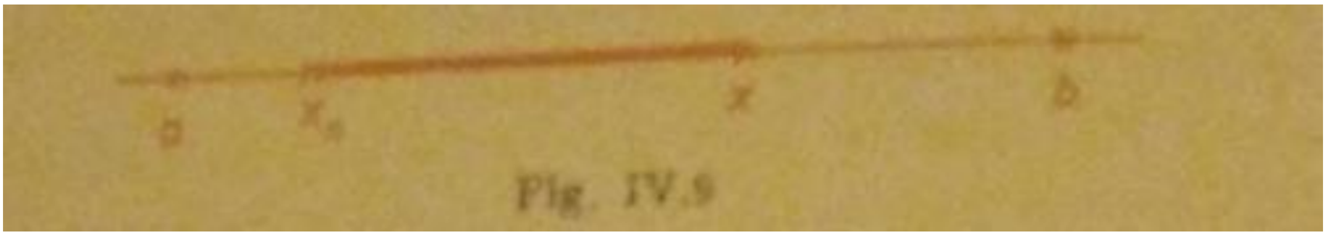


Fig. IV.9

At small intervals of length we can assume that the mass is distributed homogeneously and the idealization of this fact is taking into account the limit

$\rho(x_0) = \lim_{x \to x_0} \frac{m(x) - m(x_0)}{x - x_0}$

(assuming it exists). In this case, an important characteristic is obtained, called the linear mass density at the point x_0. So $\rho(x_0)$=m'(x_0).

Example. We assume that a material mass is distributed on the interval [0, 4] so that m(x)=x^3 +3x, then the density at the point x_0=2 is ρ(x)=m'(x)=(3x^2+3), ρ(2)=15 kg/m

2.4. Homework

1. The law of motion of a mobile on an axis is $s(t)=t^2-2t+1$, any t≥0. Calculate the speed and acceleration of the mobile at the moment t=1s. How do you explain the obtained result?

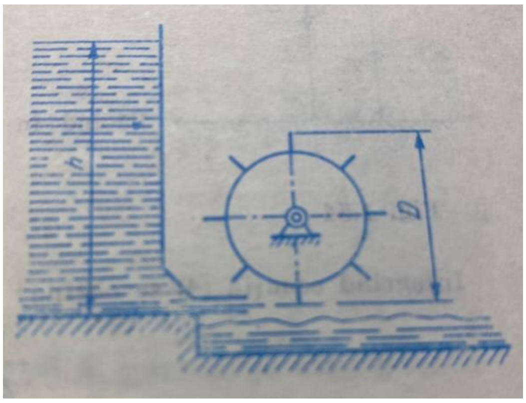
2. The law of motion of a mobile on an axis is $s(t)=e^{kt}\cdot\cos(2t)$,any t∈R. Determine the constant k knowing that s''(t)+s'(t)+5s(t)=0, any t. Then calculate the speed and acceleration of the mobile at the moments t=0, t=$\frac{\pi}{2}$ and for t→ ∞.

3. The law of motion of a mobile on an axis is $s(t)=t^3-12t^2+4$. At what moment is its acceleration 0? What is the minimum value of the speed of the mobile?

4. We assume that at each moment t the amount of energy flowing through a conductor is Q(t)=2cos(πt). Determine the intensity of the current. At what moment is the maximum intensity? And the minimum?

3. The application of the derivative to make the connection between physics, mathematics and technique

To perform an approximate calculation of the maximum power and determine the speed of a hydraulic wheel, operated from the bottom, if known: The height of the water column h, the cross section of the water jet S and the diameter of the wheel D. The stream of water continuously acts on the palettes and after it hits it falls. Numerical application: H=5m, S=0.6 m2, D=3m



Solution

The speed of the water jet when colliding with the hydraulic wheel paddle is given by the relation :

$v = \sqrt{2gh}$

We will assume that the water jet after the paddle collision continues the movement, with the speed of the v_k paddle, $v_k < v$.

In this case the mass flow $Q_m = m/\Delta t = \rho l S/\Delta t$

$Q_m = S(v-v_k)\rho_a = S(\sqrt{2gh}-v_k)\rho_a$

lose in every unit of time (v-vk)speed.

Because of this, the force $F = ma = m(v-v_k)/\Delta t$

$F = Q_m(v-v_k) = S(\sqrt{2gh}-v_k)^2\rho_a$ on the wheel is applied,

so the power of the hydraulic wheel is : $P=Fv_k$

$$P = S(\sqrt{2gh}-v_k)^2\rho_a v_k \qquad (1)$$

To determine the extremes of P=f(vk) we have:

$$\frac{dP}{dv_k} = S\,\rho_a\,(\sqrt{2gh}-v_k)\,(\sqrt{2gh}-3v_k)=0 \qquad (2)$$

of which $v_{1k} = \sqrt{2gh}$ - which is not possible because $v_k < v$

$$v_{2k} = \frac{1}{3}\sqrt{2gh} \qquad (3)$$

It has been observed since (2) that for $v_k = v_{2k}$ the function P=f(v) has a maximum

Replacing (3) in (1) is achieved

$$P_{max} = \frac{4}{27}S\rho\,(2gh)^{3/2} \qquad (4)$$

The optimum angular rotational speed of the hydraulic wheel is

$\omega = 2\frac{v_{2k}}{D} = \frac{2}{3D}\sqrt{2gh}$, and the frequency of rotation n is $\omega = 2\pi n$

$$n = \frac{2}{3\pi D}\sqrt{2gh} = \frac{2\,.60}{3\pi D}\sqrt{2gh} \cong \frac{20}{D}\sqrt{2h}\ \text{(rot/min)},\quad \pi^2 \cong 10 \qquad (5)$$

Replacing the numerics in (4) and (5) results

Pmax = 86,368kW, n≅31 rot/min

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