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Colegiul Național „Constantin Diaconovici Loga" din Timișoara

Co-funded by the Erasmus+ Programme of the European Union

Project title: **Computer supported Math teaching,** COMPMATH
Ref. No.: KA210-SCH-ABDA6A90
Action Type: KA210-SCH - Small-scale partnerships in school education

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Quartic Investigation

2022 - 2023

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Introduction

In this investigation we are going to limit ourselves to quartic functions that have exactly three turning points. We will do this by considering a particular function

$$f(x) = x^4 - 8x^3 + 18x^2 - 12x + 24.$$

Its inflection points (Q and R) will be found, and then intersection points between the line QR and the quartic function will be computed. These points will determine segments that will allow us to introduce the Golden Ratio. The same steps will be applied for other quartic functions, and a conjecture statement will be formulated. Finally, the procedure for proving the conjecture will be described.

1. Graphing the function $f(x) = x^4 - 8x^3 + 18x^2 - 12x + 24$

In order to graph the function, the Desmos graphing Calculator feature is used. The graph of the function $f(x) = x^4 - 8x^3 + 18x^2 - 12x + 24$ is presented on Figure 1.

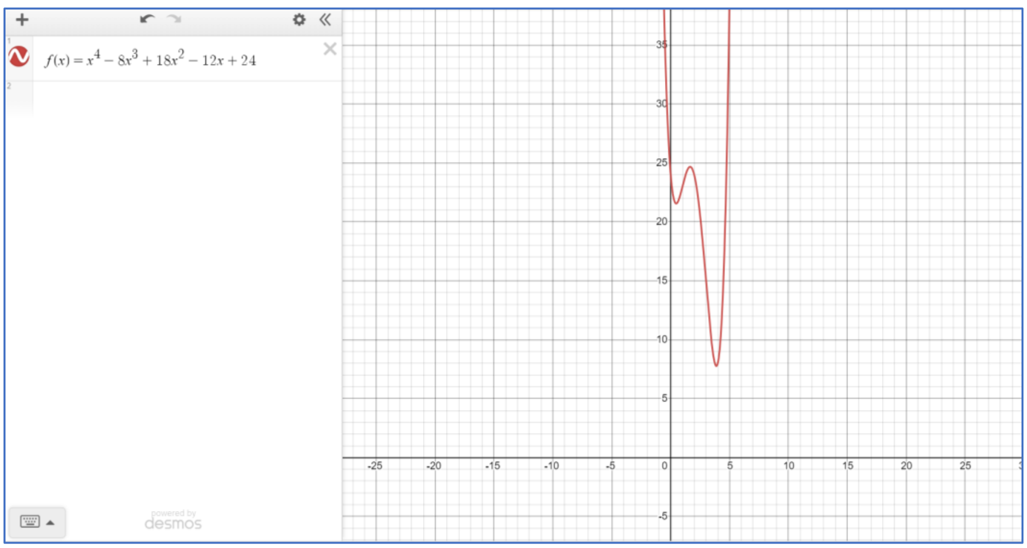


Figure 1: Graphical representation of the function $f(x) = x^4 - 8x^3 + 18x^2 - 12x + 24$

2. Finding the coordinates of Q and R the points of inflection

To find the points of inflection we need to solve the equation $f''(x) = 0$. This implies finding the second derivative of the function $f(x) = x^4 - 8x^3 + 18x^2 - 12x + 24$.

For the first derivative, the power rule was used to find $f'(x)$:

$$f'(x) = 4x^3 - 24x^2 + 36x - 12.$$

For the second derivative, the power rule was used again to find $f''(x)$:

$$f''(x) = 12x^2 - 48x + 36.$$

Then the $f''(x)$ is set to be equal to zero, $12x^2 - 48x + 36 = 0$, which is a quadratic equation having solutions $x_1 = 1$ and $x_2 = 3$, as it can be seen on Figure 2.

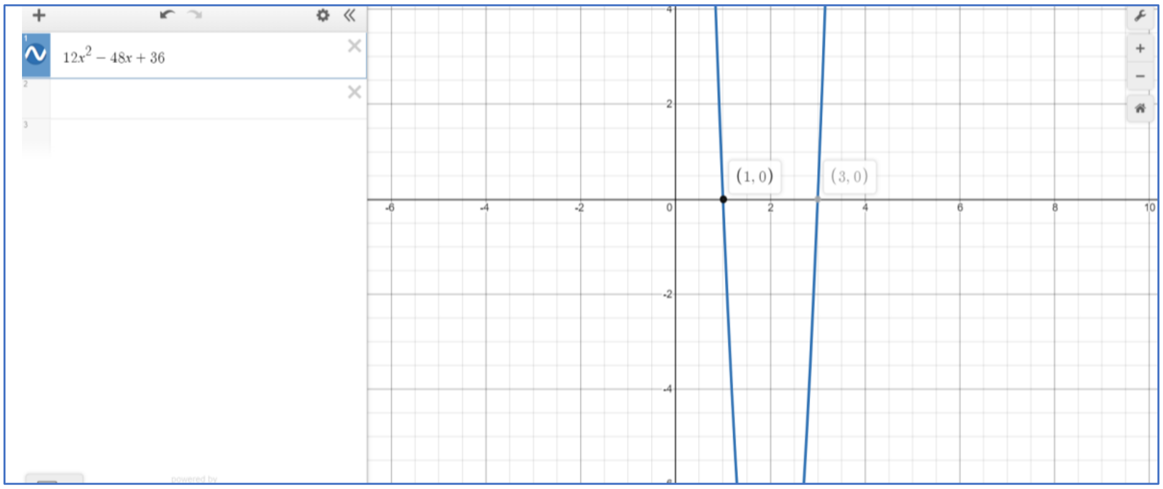


Figure 2: Zeros of the function $f''(x) = 12x^2 - 48x + 36$.

This means that there are 2 inflection points, $x_1 = 1$ and $x_2 = 3$, of the original quartic function f (x). However the corresponding y-values are f (1) and f (3). Substituting these values into the original f (x) function we found out that

f (1)	f (3)
$(1)^4 - 8(1)^3 + 18(1)^2 - 12(1) + 24 = 23$	$(3)^4 - 8(3)^3 + 18(3)^2 - 12(3) + 24 = 15$

In conclusion, the coordinates of the inflections point are Q(1 , 23) and R(3 , 15). To determine their concavity (concave up or concave down) test values have to be taken.

x	$-\infty \rightarrow 1$	$1 \rightarrow 3$	$3 \rightarrow +\infty$
$f''(x) = 12x^2 - 48x + 36$ f(x)	+ Concave up	- Concave down	+ Concave up

3. Finding intersection point between the line QR and quartic function

3.1. Determining the intersection points W, R, Q and V

To determine the straight line that intersects with f (x) we need to use the formula:

$$y - y_1 = m(x - x_1),$$

where (x_1, y_1)=(3 , 15), and the slope m is $\frac{y_2-y_1}{x_2-x_1} = \frac{15-23}{3-1}$ = -4.

Substituting the point and the slope into the equation $(y - y_1) = m(x - x_1)$, we get y-15 = -4 (x-3), that is equivalent to

$$Y = -4x + 27.$$

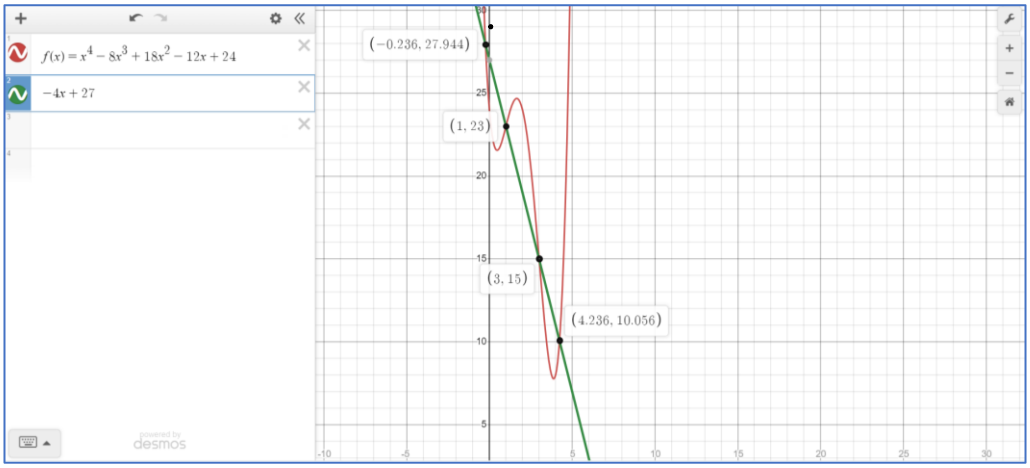


Figure 3: Intersection of the line y = -4x + 27 with the graph of the function f(x)

This graph shows us that there are 4 intersection points between the line y = -4x + 27 and the quartic function f (x) = $x^4 - 8x^3 + 18x^2 - 12x + 24$:

W (-0.236 , 27.944) , R(1 , 23) , Q (3 , 15) , and V (4.236 , 10.056).

3.2 Calculating the ratio WQ: QR: RV

To find the ratio of any point on a graph, we need to compute the distances. For that we use distance formula $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. The points

W (-0.236 , 27.944)

R (1 , 23)

Q (3 , 15)

V (4.236 , 10.056)

are used in the above distance formula, and hence the lengths of the following line segments are determined:

WQ = $\sqrt{(-0.236 - 3)^2 + (27.944 - 15)^2}$ = 13.342

QR = $\sqrt{(1-3)^2+(23-15)^2}$ = 8.246

RV = $\sqrt{(1-4.236)^2+(23-10.056)^2}$ = 13.342

Thus, the ratio WQ: QR: RV is 13.342: 8.246 : 13.342.

4. Golden ratio approach into our quartic function

4.1 Simplify the ratio so that QR = 1

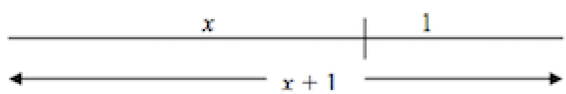
An easy way to find Golden Ratio is to divide the ratio WQ: QR: RV by 8.246, therefore making QR=1:

$$\frac{13.342}{8.246} : \frac{8.246}{8.246} : \frac{13.342}{8.246} \rightarrow 1.618 : 1 : 1.618$$

The above division shows that 1.618, namely Golden Ratio, was obtained.

4.2 Golden Ratio

The Golden Ratio is an irrational number distinguished by the Greek letter ϕ, which is approximately equal to 1.618, i.e., $\phi = \frac{1+\sqrt{5}}{2}$. Indeed, according to the Ref.1, if we are dividing a segment in such way that shorter part "1" and longer part "x" satisfy the proportion



$$\frac{1}{x} = \frac{x}{x+1},$$

then, the proportion is equivalent to the quadratic equation $x^2 - x - 1 = 0$ which has the root $\phi = \frac{1+\sqrt{5}}{2}$ and its conjugate $\frac{1-\sqrt{5}}{2}$.

In the other words, if two quantities are in Golden Ratio, their ratio to each other is the exact same as the ratio of their sum to the larger of the two quantities (see Figure 4).

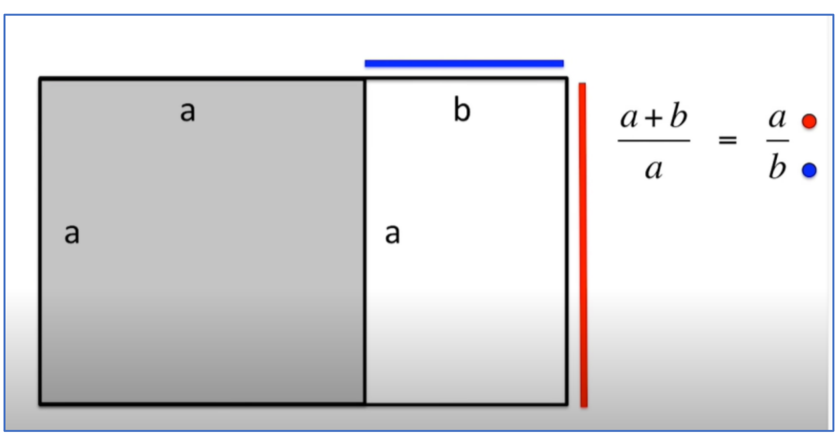


Figure 4: Illustration of the Golden Ratio

The same Golden Number 1.618 (i.e., ratio $\phi = \frac{1+\sqrt{5}}{2}$) is produced when we are dividing the Fibonacci sequences numbers one after the other (see Ref. 2). The Fibonacci sequence begins with 0 and 1; the previous two numbers are added to produce the next number in the sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34…. The Golden Number will appear soon by dividing any number from sequence with the number before it: 1:1, 2:1, 3:2, 5:3, 8:5, 13:8, 21:13, 34:21….., that is equivalent to 1, 2, 1.5, 1.666, 1.625, 1.615, 1.619…. which converges to 1.618.

The Golden Ratio was also used in the creation of ancient Greek architecture and many famous artworks. Over 500 years ago, the Renaissance thinker Luca Pacioli was also bewitched by this ratio, so he wrote the book the "De divina proportione", illustrated by Leonardo da Vinci (Ref. 3). The book studies various geometric solids and the golden ratios appearance in architecture and the human body.

In other words, the Golden Ratio can be found in architecture (Pyramids, Taj Mahal and the Ancient Greek Temples), art (Mona Lisa), proportions of human body, and even in nature itself.

5. Repeating steps 1-4 for another quartic curve with three turning points

Let $f(x) = x^4 - 2x^3 - 3x^2 + 4x - 3$ be another quartic function with three turning points.

The graph represented on the Figure 5, shows three turning points (W shape).

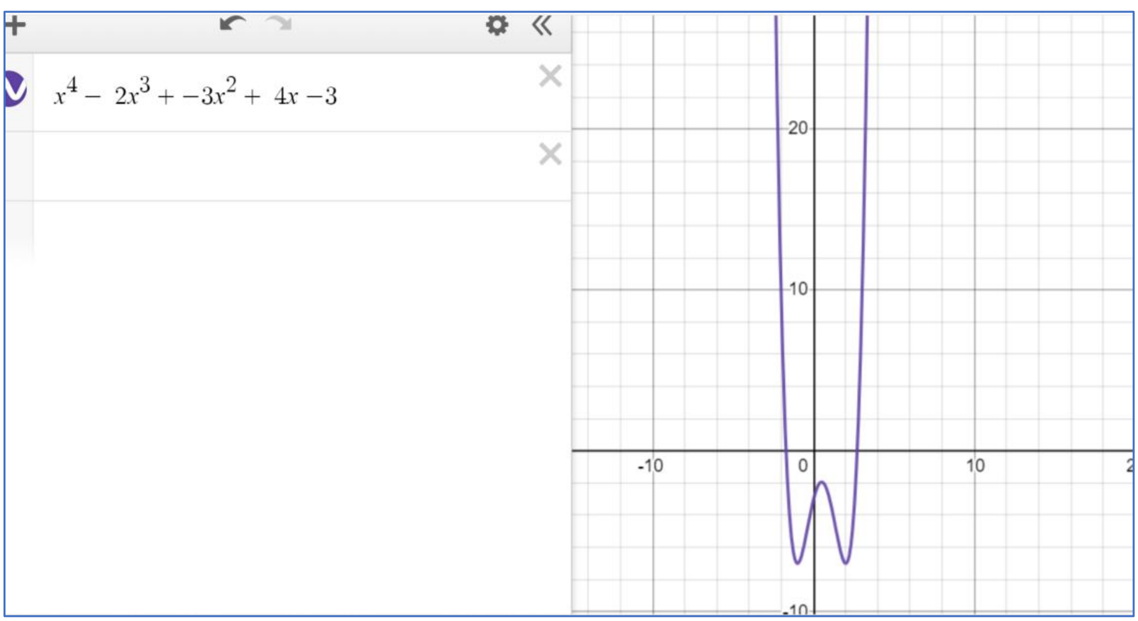


Figure 5: Graphical representation of the function $f(x) = x^4 - 2x^3 - 3x^2 + 4x - 3$

For inflection points, we are solving the equation f ″ (x) = 0. In this case, derivatives are $f'(x) = 4x^3 - 6x^2 - 6x + 4$ and $f''(x) = 12x^2 - 12x - 6$.

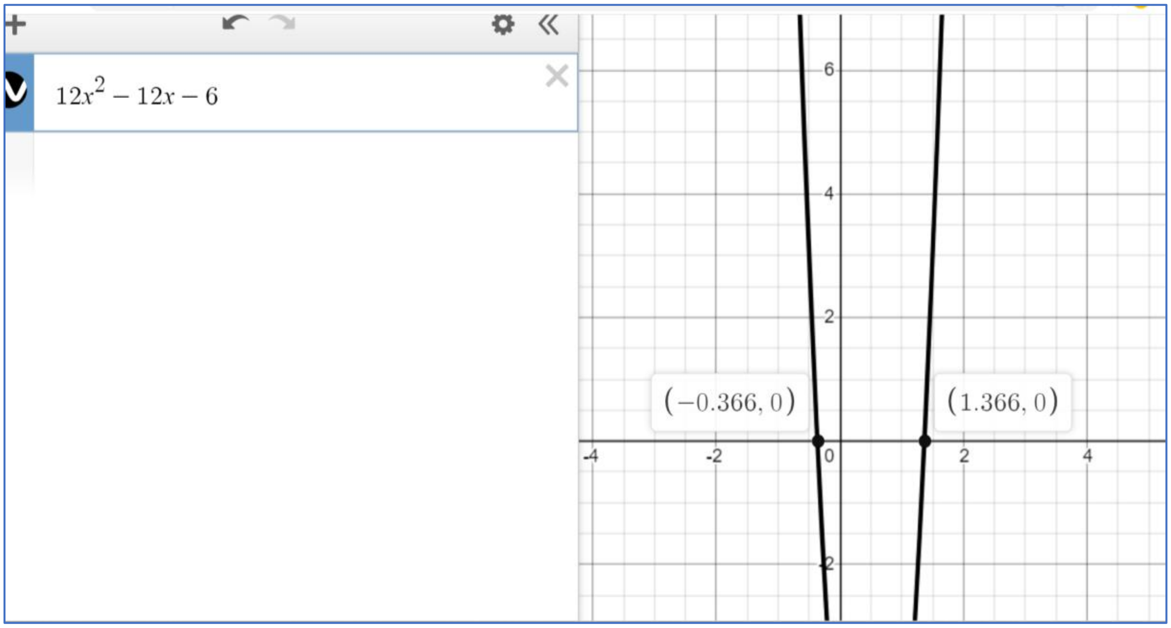


Figure 6: Zeros of the function $f''(x) = 12x^2 - 12x - 6$

The function $f(x) = x^4 - 2x^3 - 3x^2 + 4x - 3$ has two points of inflection at $x_1 = -0.366$ and $x_2 = 1.366$.

f (-0.366)	f (1.366)
$(-0.366)^4 - 2(-0.366)^3 - 3(-0.366)^2 + 4(0.366) - 3 = -4.79$	$(1.366)^4 - 2(1.366)^3 - 3(1.366)^2 + 4(1.366) - 3 = -4.75$

The coordinates of the inflection points are R(-0.366 , -4.79) and Q(1.366 , -4.75). To determine their concavity (concave up or concave down) test values have been taken.

x	$-\infty \rightarrow -0.366$	$-0.366 \rightarrow 1.366$	$1.366 \rightarrow +\infty$
$f''(x) = 12x^2 - 12x - 6$ f(x)	+ Concave up	- Concave down	+ Concave up

To determine a straight line QR that intersects graph of the function f (x) we use the formula

$$y - y_1 = m(x - x_1)$$

that is equivalent to y + 4.75 = m (x - 1.366), where slope *m* is $\frac{y_2-y_1}{x_2-x_1} = \frac{-4.75+4.79}{1.366+0.366} = 0.0231$.

Inputting everything into the equation $(y - y_1) = m(x - x_1)$, the equation line is obtained

$$y + 4.75 = 0.0231(x - 1.366)$$

that means

$$y = 0.0231x – 4.78.$$

The intersection between this line and the graph of the function f(x) is represented on the Figure 7.

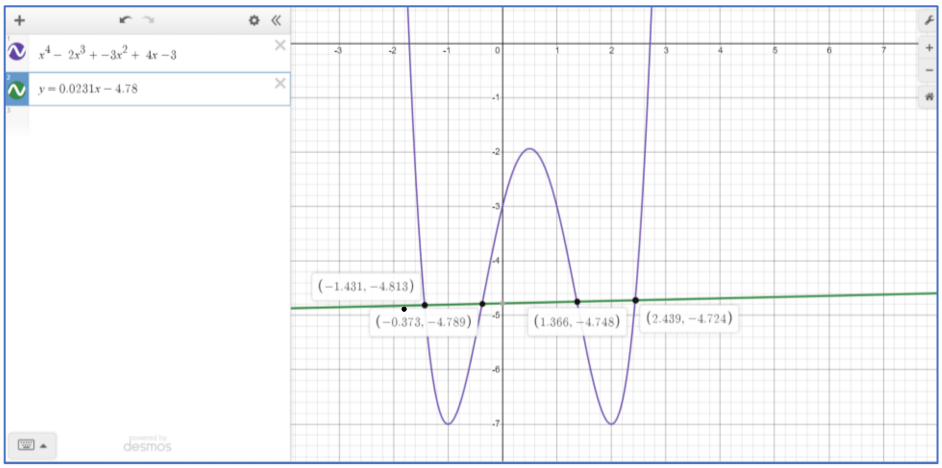


Figure 7: Intersection of the line y= 0.0231x – 4.78 and graph of the function

$f(x) = x^4 - 2x^3 - 3x^2 + 4x - 3$

Coordinates of the intersection points are:

W(-1.431 , - 4.813) , R(- 0.373 , - 4.789), Q(1.366 , -4.748) , and V(2.439 , -4.724).

A small error propagation is observed for the inflection point R, if we compare it with previously value $x_R = -0.366$. This error will be seen in the following calculations for the golden number.

We will now find the distances, i.e. the lengths of the segments necessary for the ratio WQ: QR: RV. Using the distance formula $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, we obtain:

WQ = $\sqrt{(-1.431 - 1.366)^2 + (-4.813 + 4.748)^2}$ = 2.798 (a small error propagation)

QR = $\sqrt{(- 0.373 - 1.366)^2 + (- 4.789 + 4.748)^2}$ = 1.739

RV = $\sqrt{(-0.373 - 2.439)^2 + (-4.789 + 4.724)^2}$ = 2.813

Ratio WQ: QR: RV is 2.8 : 1.74 : 2.82.

The ratio of QR has to be equal to 1, therefore we divide every ratio by 1.739

$$\frac{2.798}{1.739} : \frac{1.739}{1.739} : \frac{2.813}{1.739} \rightarrow 1.608 : 1 : 1.618.$$

and the golden number is again obtained (with a small propagation error in the third digit).

6. Conjecture statement and technology use to test different quartic functions

Let f(x) be a quartic function with three turning points, and two inflection points having abscissas x_1 and x_2. If the line determined by its inflection points intersects the quartic function into four points $A_0(x_0, f(x_0))$, $A_1 (x_1, f(x_1))$, $A_2 (x_2, f(x_2))$, and $A_3 (x_3, f(x_3))$, where $x_0 < x_1 < x_2 < x_3$, then the length of the segments A_0A_2, A_1A_2, and A_1A_3 yield the Golden Ratio as follows:

$$\frac{A_0A_2}{A_1A_2} = \frac{A_1A_3}{A_1A_2} = 1.618$$

Test the conjecture:

We will consider a special quartic function $f(x) = x^4 - 6x^2 + 5$ with three turning points and inflection situated on the horizontal axis.

- Inflection points: $f''(x) = 0 \Rightarrow 12x^2 - 12 = 0 \Rightarrow x_1 = -1$ and $x_2 = 1$.
  Thus, inflection points have coordinates $A_1 (-1, 0)$, $A_2 (1, 0)$.
- Line determined by the points $A_1 (-1, 0)$, $A_2 (1, 0)$ is the horizontal axis y=0.
- Intersection points between the line A_1A_2 and graph of the function $f(x) = x^4 - 6x^2 + 5$ are:

$$A_0\left(-\sqrt{5}, 0\right), \qquad A_1(-1, 0), \qquad A_2(1, 0), \qquad A_3\left(\sqrt{5}, 0\right).$$

- The segments lengths are:

$$A_0A_2 = 1 + \sqrt{5}, \quad A_1A_2 = 2, \;\; A_1A_3 = 1 + \sqrt{5}.$$

- Golden Ratios are:

$$\frac{A_0A_2}{A_1A_2} = \frac{1+\sqrt{5}}{2} = 1.618 \quad \text{and} \quad \frac{A_1A_3}{A_1A_2} = \frac{1+\sqrt{5}}{2} = 1.618$$

Use technology:

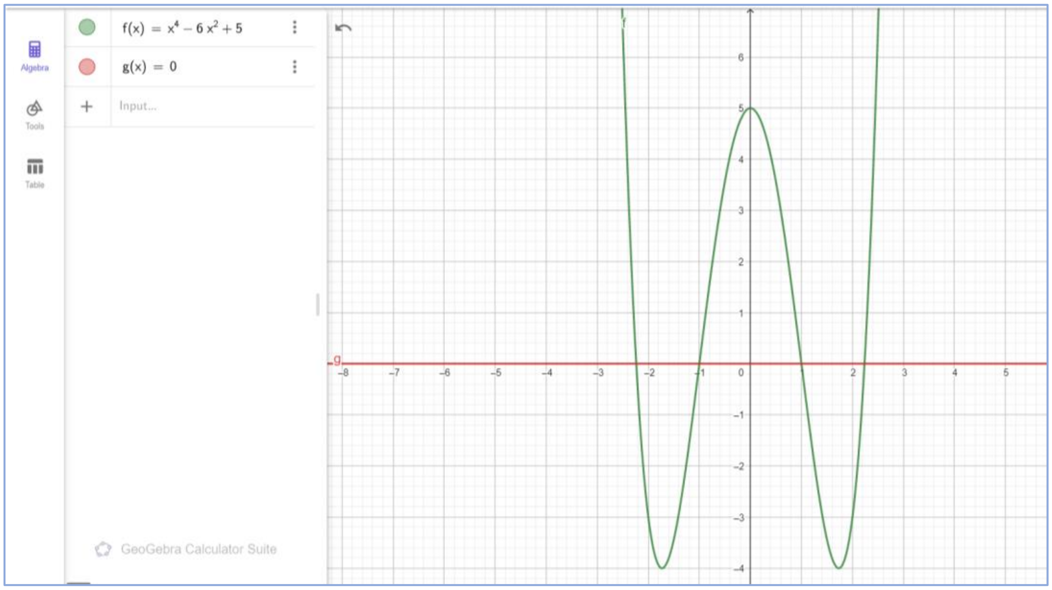


Figure 8: Intersection of the line y= 0 and graph of the function f(x) = $x^4 - 6x^2 + 5$

7. Procedure to prove the conjecture

Let $f(x) = m_4x^4 + m_3x^3 + m_2x^2 + m_1x + m_0$ be a quartic function with three turning points, and two inflection points.

Step 1: Solve the equation $f''(x) = 0$ and find abscissas x_1, x_2 of the inflection points.

Step 2: Determine the equation of the line determined by the inflection points:

$$y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$

Step 3: Determine the abscissas x_0, x_3 of the intersection points between the above line and the graph of f(x), where $x_0 < x_1 < x_2 < x_3$.This can be performed graphically or by solving the equation:

$$m_4x^4 + m_3x^3 + m_2x^2 + m_1x + m_0 = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$

Step 4: Determine the points $A_0(x_0, f(x_0))$, $A_1(x_1, f(x_1))$, $A_2(x_2, f(x_2))$, and $A_3(x_3, f(x_3))$

Step 5: Compute the distances A_0A_2, A_1A_2, and A_1A_3 and check the Golden Ratio

$$\frac{A_0A_2}{A_1A_2} = \frac{A_1A_3}{A_1A_2} = 1.618$$

8. Conclusion

In conclusion, if we consider a quartic function f(x) with 3 turning points (the first derivative changes the sign), and two inflection points (second derivative has two roots x_1, x_2), then the line determined by inflection points will intersect the graph of f(x) in four points, namely $A_0(x_0, f(x_0))$, $A_1 (x_1, f(x_1))$, $A_2 (x_2, f(x_2))$, and $A_3 (x_3, f(x_3))$, where $x_0 < x_1 < x_2 < x_3$. With these points we can always form three ratios that satisfy Golden Ratio, for example:

$$\frac{A_0A_2}{A_1A_2} = \frac{A_1A_3}{A_1A_2} = 1.618$$

Extension: Alternative method to prove conjecture for all quartic functions

This conjecture can also be proven using the following method:

Let $f(x) = m_4x^4 + m_3x^3 + m_2x^2 + m_1x + m_0$ be a polynomial function of degree 4 with inflection points $x_1=b$, $x_2=c$.

Step 1: Consider the function $f''(x)$ with roots b and c:

$$f''(x)= 12m_4(x - b)(x - c) = 12m_4x^2 -12m_4 (b + c)x + 12m_4 bc.$$

Step 2: Recover the polynomial function f(x) by integrating $f''(x)$ twice and letting m_0 and m_1 be the constants for integration. We get

$$\int f''(x)\, dx = 4m_4x^3 - 6m_4 (b + c)x^2 + 12m_4bcx + m_1 = f'(x)$$

$$\iint f'(x)\, dx = m_4x^4 - 2m_4 (b + c)x^3 + 6m_4bcx^2 + m_1x + m_0 = f (x)$$

Step 3: Write the equation of the line through the inflection points $x_1=b$, $x_2=c$:

$$y = \frac{f(c)-f(b)}{c-b}\,(x-b) + f(b),$$

from where

y =

$$\frac{[4m_4(c)^4-2m_4\ (b+c)(c)^3+6m_4bc(c)^3+m_1c+m_0]-[4m_4(b)^4-2m_4\ (b+c)(b)^3+6m_4bc(b)^2+m_1b+m_0]}{c-b}(x-b)+$$

$$+[4m_4(b)^4 - 2m_4\ (b+c)(b)^3 + 6m_4bc(b)^2 + m_1b + m_0\ ]$$

Step 4: After equating y(x) = f(x) to find the 4 intersection points, calculations prove that two solutions are exactly inflection points $x_1 = b$, $x_2 = c$, and the other two roots will be:

$$x_0 = \frac{1+\sqrt{5}}{2}x_1 + \frac{1-\sqrt{5}}{2}x_2 = \phi b + \phi^* c$$

$$x_3 = \frac{1+\sqrt{5}}{2}x_2 + \frac{1-\sqrt{5}}{2}x_1 = \phi c + \phi^* b,$$

where $x_0 < x_1 < x_2 < x_3$.

Therefore the golden number $\phi = \frac{1+\sqrt{5}}{2}$ is obtained, together with its conjugate $\phi^* = \frac{1-\sqrt{5}}{2}$, and the conjecture is proven.

References:

[1] McMullin, Lin (2006) “The Golden Ratio and Quartic Polynomials.” https://teachingcalculus.files.wordpress.com/2016/01/golden_ratio_in_quartics_2007.pdf

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[4] White, Peirce. “4.2 Polynomial Functions and Models. A Polynomial Function Is a Function of the Form. - Ppt Download.” *SlidePlayer*, Pierce White, Retrieved from https://slideplayer.com/slide/9549740/